Key

Birzeit University
Mathematics Department

Math 330 - First Exam
First Semester 2013/2014

StudentName:..Number:.......................Section:........

Question 1. (16 points) *Consider the function:* $g(x) = \sqrt[3]{\cos x + 1}$

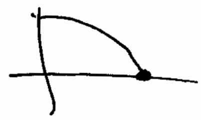
a) Show that $g(x)$ *has a fixed point in* $[0, \frac{\pi}{2}]$.

b) Show that the fixed point iteration converges for any p_0 *in* $[0, \frac{\pi}{2}]$.

c) let $p_0 = 1$. *Estimate the fixed point of* $g(x)$ *with error less than* 10^{-2}.

d) Find the number of iterations needed to get Accuracy 10^{-2} *(theoretically).*

(a) Using the graph of $\cos x$ on $[0, \pi/2]$



it is decreasing on $[0, \pi/2]$

2 so is $\sqrt[3]{\cos x + 1} = g(x)$

and since $g(0) = 1.2599 \in [0, \pi/2]$

and $g(1) = 1 \in [0, \pi/2]$

2 so $g(x) \subset [0, \pi/2]$ for every $x \in [0, \frac{\pi}{2}]$

$\Longrightarrow$ $g(x)$ has a fixed point in $[0, \pi/2]$

(b) $|g'(x)| = \left| \dfrac{-\sin x}{3\sqrt[3]{(\cos x+1)^2}} \right| < \dfrac{1}{3\sqrt[3]{(\cos x+1)^2}}$ in $[0, \pi/2]$

2 since $\sqrt[3]{(\cos x+1)^2}$ is decreasing in $(0, \pi/2)$

so $\dfrac{1}{3\sqrt[3]{(\cos x+1)^2}}$ is increasing so

2 $|g'(x)| = \dfrac{1}{3\sqrt[3]{(\cos x+1)^2}} \leq |g'(\frac{\pi}{2})| = \dfrac{1}{3} < 1$ in $[0, \pi/2]$

so $g(x)$ has a unique fixed point in $[0, \pi/2]$

and the FPI converges $\forall p_0 \in [0, \frac{\pi}{2}]$

(c) If $P_0 = 1$, $P_{n+1} = g(P_n)$

$\Rightarrow$ $P_1 = 1.15488$

$P_2 = 1.11976$

(d) Using $k = 1/3$ +

$$\frac{k^n |P_1 - P_0|}{1 - k} \leq 10^{-2}$$

$$\frac{\left(\frac{1}{3}\right)^n (0.15488)}{\frac{2}{3}} < 10^{-2}$$

$n \geq 2.86$

$n = 3$

Question 2. (13 points) *Let* $f(x) = x^3 - \cos x - 1$

a) Use Newton-Raphson method to estimate the root of the above function in $[0, \frac{\pi}{2}]$ *starting with* $p_0 = 1$ *and with error* $\leq 10^{-3}$.

b) What do you expect the order of convergence of the above iteration, prove your claim numerically ?

(a) $g(x) = x - \dfrac{f(x)}{f'(x)}$

$= x - \dfrac{x^3 - \cos x - 1}{3x^2 + \sin x}$

$P_{n+1} = g(P_n)$

$P_0 = 1$ (2)

$P_1 = 1.11622 1633$ (2)

$P_2 = 1.12649 0307$ (2)

$P_3 = 1.12656 1905$

(b) Expect $R = 2$ (2) quadratic convergence

and $A \approx \dfrac{f''(p)}{2 f'(p)} = 0.76$ (2)

Estimating the error by $|P_{k+1} - P_k| \approx e_k$

k	$\dfrac{\lvert e_{k+1}\rvert}{\lvert e_k\rvert^2}$
1	0.76022
2	0.679 ..

(3)

$\downarrow$ 0.76

Question 3. (7 points) *Use Newton's method to find x^1 for the following system:*

$$3x^2 = \cos y$$

$$3ye^x - e^y = 1$$

Given that $x^0 = (1,1)^t$

$f_1(x,y) = 3x^2 - \cos y = 0$

$f_2(x,y) = 3ye^x - e^y - 1 = 0$

$f_1(1,1) = 2.460$

$f_2(1,1) = 4.436$

2

$$J = \begin{pmatrix} 6x & \sin y \\ 3ye^x & 3e^x - e^y \end{pmatrix}\Bigg|_{(1,1)}$$

$$= \begin{pmatrix} 6 & 0.84 \\ 8.15 & 5.44 \end{pmatrix}$$

3

Solving

$$-\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\Bigg|_{(1,1)} = J\Big|_{(1,1)} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$\Delta x = -0.374$

$p_1 = \Delta x + p_0 = 0.626$

$\Delta y = -0.255$

$q_1 = 0.745$

Question 4. (7 points)*Find the total cost of Solving* 5×5 *system using LU- Factorization method.*

step	+/−	×/÷
1	4×4	4 + 4×4
2	3×3	3 + 3×3
3	2×2	2 + 2×2
4	1	1+1
total	30	10 + 30

70

Forward 20

Backward 25

115

Question 5. (7 points) *A small Business has weekly Profits of*

$$P(x) = x^2 + 4x - 2e^x$$

where x is the number of hundreds of units produced each week. approximate (with error less than 10^{-2})the level of production that yields the Maximum Profit

$f(x) = P'(x) = 2x + 4 - 2e^x = 0$

2

$f(0) = 2$

$f(2) < 0$

The root ~~lies~~ in $[0, 2]$

Af $p_0 = 1$

$p_1 = 1.1639$

$p_2 = 1.1464$

$p_3 = 1.1462$

5

$x = 1.1462$ (in hundred)